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Are dualities appropriate for duality theories in optimization?

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Abstract We raise some questions about duality theories in global optimization. The main one concerns the possibility to extend the use of conjugacies to general dualities for studying dual optimization problems. In fact, we examine whether dualities are the most general concepts to get duality results. We also consider the passage from a Lagrangian approach to a perturbational approach and the reverse passage in the framework of general dualities. Since a notion of subdifferential can be defined for any duality, it is natural to examine whether the familiar interpretation of multipliers as generalized derivatives of the performance function associated with a dualizing parameterization of the given problem is still valid in the general framework of dualities.

Keywords Conjugacy · Duality · Lagrangian · Mathematical programming · Multiplier · Optimization · Performance function · Perturbation · Subdifferential

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1 Introduction

Duality is a general scheme which goes beyond optimization theory and practice. It can be roughly described as the introduction of an associated problem (the dual problem) to a given one (the primal problem) in order to draw some information or some help for solving the primal problem. In optimization, the information one may get may be useful for a numerical analysis (see [6,11] for instance); it is often under the form of an estimate of the value of the problem. Adopting the viewpoint of unilateral analysis, we showed in Refs. [68,61] that one-sided concepts of Lagrangians and perturbations could be used to get such estimates.

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However, our study was limited to the use of conjugacies for which the devices are close to the usual ones in convex duality (see [5,40,46,47,51,62,63,81,86]).

It is the purpose of the present article to examine the possibility of using dualities in the sense of Refs. [45,54,81] or in the broad setting of lattices ([12,42,50,54,87]) in duality theories for optimization. Since the Lagrangian theory essentially consists in an exchange of order between a minimization and a maximization, and thus does not involve any sophisticated apparatus, we mainly focus our attention on the theory of perturbations. Embedding a given problem into a family of parametrized problem is often natural and fruitful in terms of sensibility analysis. Then the relation linking the value (or performance) function p to its biconjugate is the key to what is called the weak duality inequality. When the conjugate is defined with the help of a conjucacy, the maximization of the opposite of the conjugate of p defines a dual problem. Because dualities between lattices of functions can be defined through generating functions, a dual problem also arises in such a framework. However, a number of "dualities" considered in the literature are not dualities in the classical sense evoked above (see [1,9,15,66]...). In Ref. [54] they are called "quasi-dualities" because they can also be defined by a generating function. Thus, it is tempting to consider duality theories in the framework of quasi-dualities. Such an apparatus allows to define a dual problem, and even a bidual problem and to give some links with a Lagrangian approach. However, weak duality is no more an inequality. Thus, we rather limit our study to the case of what we call "pseudo-dualities". They consist in the addition of a reverse mapping to the given quasi-duality, ensuring an appropriate inequality. The choice of such a mapping may differ; but when a duality is given, there is a natural one, the reverse duality.

Our general scheme can be applied to the constrained optimization problem

$$(\mathcal{P}) \quad \text{minimize } f(x) \text{ subject to } x \in F.$$

The decision space X has not to be endowed with any special structure and the feasible set $F \subset X$ can be any subset. Also, the objective function $f : X \to \mathbb{R}_\infty := \mathbb{R} \cup \{\infty\}$ may be an arbitrary function. When F is the value at some base point 0_Z of some parameter space Z of a multimapping $\Phi : Z \rightrightarrows X$, a natural perturbation of problem $(\mathcal{P})$ is available and a duality theory can be applied provided one disposes of a duality between $\overline{\mathbb{R}}^Z$ and some other function space $\overline{\mathbb{R}}^Y$. Such a case contains many special situations, in particular mathematical programming problems.

In Sect. 2, some concepts and results in the line of the monographs [51,74,82] about general dualities are recalled and completed. Applications to the Lagrangian and perturbational theories are described in Sects. 3 and 4, respectively. Passages between the two approaches are given in Sect. 5. For the convex case, which serves as a model for the whole theory, we refer to Refs. [3,4,19,35,47,70,90] for example.

Numerous contributions ([2,13,27,30,31,58,62–67,72–79]...) show the interest of dealing with dualities in a non classical framework and some examples are displayed in Sect. 7. We hope the present paper will contribute to the usefulness of such a general approach.

2 Dualities and their relatives in optimization

A general notion of duality has been given in Refs. [12,42,50]. It takes place in complete lattices, or even in general ordered spaces.

Definition 1 Given ordered (or pre-ordered) sets $\mathcal{Y}$ and $\mathcal{Z}$, a *duality* between $\mathcal{Y}$ and $\mathcal{Z}$ is a mapping $D : \mathcal{Y} \to \mathcal{Z}$ noted $f \mapsto f^D$ such that

$$D\left(\bigwedge_{i\in I} f_i\right) = \bigvee_{i\in I} D(f_i) \tag{1}$$

for any family $(f_i)_{i\in I}$ in $\mathcal{Y}$ for which $\bigwedge_{i\in I} f_i := \inf_{i\in I} f_i$ exists.

The sets $\mathcal{Y}$ and $\mathcal{Z}$ are often subsets of the spaces $\overline{\mathbb{R}}^Z$ and $\overline{\mathbb{R}}^Y$ of extended real-valued functions on sets Z and Y, respectively. The case in which $\overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty, +\infty\}$ is replaced by a sublattice such as $\overline{\mathbb{R}}_+$ or $\{0, 1\}$ is also of interest. When $\mathcal{Y}$ and $\mathcal{Z}$ are the power sets $\mathcal{P}(Y)$ and $\mathcal{P}(Z)$ of sets Y and Z, respectively, or some subpaces with the induced orders, we say that $P : \mathcal{Y} \to \mathcal{Z}$ is a *polarity* if it satisfies (1) when the orders in $\mathcal{Y}$ and $\mathcal{Z}$ are the reverse orders of inclusion, i.e. if for any family $(A_i)_{i\in I}$ of subsets of Y one has

$$P\left(\bigcup_{i\in I} A_i\right) = \bigcap_{i\in I} P(A_i). \tag{2}$$

Using the *indicator function* ι_S of a subset S of a set X given by $\iota_S(x) = 0$ if $x \in S$, $+\infty$ else and considering the injections $A \mapsto \iota_A$ and $B \mapsto \iota_B$ of $\mathcal{P}(Y)$ and $\mathcal{P}(Z)$ into $\overline{\mathbb{R}}^Y$ and $\overline{\mathbb{R}}^Z$, respectively as identifications, any polarity can be considered as a special duality between sublattices of some function spaces.

Polarities abound. They generalize orthogonality. They can be defined by a coupling function $c : Y \times Z \to \overline{\mathbb{R}}$ and a fixed real number r by setting

$$P(A) := \{z \in Z : \forall y \in A\ c(y, z) \leq r\}.$$

For instance, the monotone polarity $A \mapsto A^\mu$ from $\mathcal{P}(Y)$ to $\mathcal{P}(Z)$, with $Y := U \times V$, $Z := U \times V$, where U and V are two spaces paired by a pairing $b : U \times V \to \mathbb{R}$, has been introduced by Martínez-Legaz by taking $r := 0$ and

$$c((u, v), (u', v')) := b(u, v') + b(u', v) - b(u, v) - b(u', v'). \tag{3}$$

A more general way of constructing a polarity consists in taking $r = -1$, a subset W of $Y \times Z$ and setting $c(y, z) := -\iota_W(y, z)$, so that

$$P(A) = \{z \in Z :\ \forall y \in A\ (y, z) \in (Y \times Z)\backslash W\}. \tag{4}$$

The most studied examples of dualities are conjugacies ([47]): given two sets Y, Z and a coupling function $c : Y \times Z \to \overline{\mathbb{R}}$, the *conjugacy* associated with c is the map $D : \mathcal{Y} := \overline{\mathbb{R}}^Y \to \mathcal{Z} := \overline{\mathbb{R}}^Z$ given by

$$f^c(z) := -\inf_{y\in Y}[f(y) - c(y, z)] = \sup_{y\in Y} -[f(y) - c(y, z)] \quad f \in \mathcal{Y},\ z \in Z.$$

Conjugacies have been characterized as dualities $D : \overline{\mathbb{R}}^Y \to \overline{\mathbb{R}}^Z$ for which the relation $D(f + r) = -(r - D(f))$ holds for every $f \in \overline{\mathbb{R}}^Y$, $r \in \overline{\mathbb{R}}$, where the addition of $\mathbb{R}$ is extended to $\overline{\mathbb{R}}$ by setting $r + (+\infty) = +\infty$ for all $r \in \overline{\mathbb{R}}$ and $(-\infty) + (-\infty) = -\infty$ and where $r - s := r + (-s)$; see [47,81].

Let us raise the question: is the notion of duality the most general concept to get duality results for optimization problems? Part of the sequel is devoted to give an answer to that question by using a weaker concept introduced in Ref. [54].

Definition 2 Given ordered (or pre-ordered) sets $\mathcal{Y}$ and $\mathcal{Z}$, a *hemi-duality* between $\mathcal{Y}$ and $\mathcal{Z}$ is a pair (D, D') of antitone mappings $D : f \mapsto f^D$, $D' : g \mapsto g^{D'}$ from $\mathcal{Y}$ into $\mathcal{Z}$ and from $\mathcal{Z}$ into $\mathcal{Y}$, respectively such that $f^{DD'} := (f^D)^{D'} \leq f$ for each $f \in \mathcal{Y}$.

Recall that D is *antitone* if $f_1^D \geq f_2^D$ whenever $f_1 \leq f_2$ and observe that $D'D$ is *homotone,* i.e. $f_1^{DD'} \leq f_2^{DD'}$ when $f_1 \leq f_2$. If for each $g \in \mathcal{Z}$ the relation $g^{D'D} := (g^{D'})^D \leq g$ also holds, one says that (D, D') is a *Galois correspondence* or a *Galois connexion* (see [12,50]); in that case the situation is entirely symmetric. When a pair (D, D') is a Galois correspondence between the complete lattices $\mathcal{Y}$ and $\mathcal{Z}$, one can draw important consequences:

$$\begin{aligned} &\forall f \in \mathcal{Y} \quad f^{DD'D} = f^D, \quad \forall g \in \mathcal{Z} \quad g^{D'DD'} = g^{D'} \\ &\forall f \in \mathcal{Y} \ \ \forall g \in \mathcal{Z} \quad \left(f^D \leq g\right) \Longleftrightarrow \left(g^{D'} \leq f\right) \\ &\forall f \in \mathcal{Y} \quad \left(f^{DD'} = f\right) \Leftrightarrow \left(\exists g \in \mathcal{Z} : \ f = g^{D'}\right). \end{aligned}$$

Thus $D'D$ and DD' are closure operations.

Hemi-dualities are easy to obtain, as shown in the next proposition which completes [54, Lemma 3.2]. Here $\mathcal{Y}$ is said to be a *complete inf-lattice* if any (nonempty) family in $\mathcal{Y}$ has an infimum. A subset $\mathcal{W}$ of an ordered space $\mathcal{Z}$ is said to be inf-cofinal if for all $z \in \mathcal{Z}$ there exists some $w \in \mathcal{W}$ such that $w \leq z$.

Proposition 3 *Given ordered spaces $\mathcal{Y}, \mathcal{Z}$ such that $\mathcal{Y}$ is a complete inf-lattice and an antitone map $D : \mathcal{Y} \to \mathcal{Z}$ such that $D(\mathcal{Y})$ is inf-cofinal in $\mathcal{Z}$, there is a greatest antitone map $D' : \mathcal{Z} \to \mathcal{Y}$ such that (D, D') is an hemi-duality, i.e. is such that $D'(D(f)) \leq f$ for every $f \in \mathcal{Y}$. It is given by*

$$D'(g) = \bigwedge\{f \in \mathcal{Y} : D(f) \leq g\} \qquad g \in \mathcal{Z}. \tag{5}$$

For any map $E : \mathcal{Z} \to \mathcal{Y}$ such that $D(E(g)) \leq g$ for all $g \in \mathcal{Z}$, one has $D' \leq E$.

If D is a duality, D' is a duality and (D', D) is a hemi-duality, i.e. one has $D(D'(g)) \leq g$ for every $g \in \mathcal{Z}$. Then (D', D), (D, D') are Galois correspondences and D' is the unique antitone map $E : \mathcal{Z} \to \mathcal{Y}$ such that $E(D(f)) \leq f$ for every $f \in \mathcal{Y}$ and $D(E(g)) \leq g$ for all $g \in \mathcal{Z}$.

Proof Clearly, D' given by (3) is well defined, antitone and such that $D'(D(f)) \leq f$ for each $f \in \mathcal{Y}$, so that (D, D') is a hemi-duality. Moreover, if $E : \mathcal{Z} \to \mathcal{Y}$ is an antitone map such that $E \circ D \leq I_{\mathcal{Y}}$, for any $g \in \mathcal{Z}$ and any $f \in \mathcal{Y}$ such that $D(f) \leq g$ one has $E(g) \leq E(D(f)) \leq f$, so that $E(g) \leq D'(g)$.

Given an arbitrary map $E : \mathcal{Z} \to \mathcal{Y}$ such that $D \circ E \leq I_{\mathcal{Z}}$, given $g \in \mathcal{Z}$, setting $f := E(g)$, we see that $D(f) \leq g$, hence $D'(g) \leq f$ by defn of D'. Since $g \in \mathcal{Z}$ is arbitrary, that proves that $D' \leq E$.

If D is a duality, for every $g \in \mathcal{Z}$, by construction of D', $D(D'(g))$ is the supremum of the family $\{D(f) : f \in \mathcal{Y},\ D(f) \leq g\}$, so that $D(D'(g)) \leq g$. Moreover, if $g = \inf_{i \in I} g_i$, then one has $D'(g) \geq D'(g_i)$ for all $i \in I$ and if f is a majorant of the family $(D'(g_i))_{i \in I}$ one has $D(f) \leq D(D'(g_i)) \leq g_i$ for all $i \in I$, hence $D(f) \leq g$ and $D'(g) \leq f$ by defn of D'. Thus $D'(g) = \sup_{i \in I} D'(g_i) : D'$ is a duality.

The uniqueness assertion stems from the inequalities $E \leq D'$ and $D' \leq E$ for any antitone map $E : \mathcal{Z} \to \mathcal{Y}$ such that $E(D(f)) \leq f$ for every $f \in \mathcal{Y}$ and $D(E(g)) \leq g$ for all $g \in \mathcal{Z}$. □

It is a classical result that a converse of the last assertion holds: for any Galois correspondence (D, D') between complete lattices, the maps D and D' are dualities.

In the sequel, we restrict our attention to lattices which are functions spaces. Not all antitone maps between functions spaces, or even dualities, are conjugacies. In order to find a substitute to the coupling function c, inspired by several examples and the contributions of Martínez-Legaz and Singer ([42–45]) we introduced in Ref. [54] the following concept in which Y and Z are arbitrary sets and the orders on $\overline{\mathbb{R}}^Y$ and $\overline{\mathbb{R}}^Z$ are the pointwise orders.

Definition 4 An antitone mapping $D : \overline{\mathbb{R}}^Y \to \overline{\mathbb{R}}^Z$, $f \mapsto f^D$, is said to be a quasi-duality if there exists a function $G : Y \times Z \times \overline{\mathbb{R}} \to \overline{\mathbb{R}}$, called a generating function, which is nonincreasing in its third variable, which satisfies $G(y, z, +\infty) = -\infty$ for all $(y, z) \in Y \times Z$ and is such that, for any $f \in \overline{\mathbb{R}}^Y$, $z \in Z$

$$f^D(z) = \sup_{y \in Y} G(y, z, f(y)). \tag{6}$$

It has been observed in Refs. [42,45] that any duality $D : \overline{\mathbb{R}}^Y \to \overline{\mathbb{R}}^Z$ has a generating function $G : Y \times Z \times \overline{\mathbb{R}} \to \overline{\mathbb{R}}$, so that it is a quasi-duality. The generating function associated with a duality D is given for $(y, z, r) \in Y \times Z \times \overline{\mathbb{R}}$ by

$$G(y, z, r) := (\iota_{\{y\}} + r)^D(z), \tag{7}$$

where $\iota_{\{y\}}$ is the indicator function of the singleton $\{y\}$. In fact, for any $f \in \overline{\mathbb{R}}^Y$, setting $f_y(\cdot) := \iota_{\{y\}}(\cdot) + f(y)$ for $y \in Y$, one has $f = \inf_{y \in Y} f_y$, hence

$$f^D(z) = \sup_{y \in Y} \left(f_y\right)^D(z) = \sup_{y \in Y} G(y, z, f(y)).$$

Note that, when D is a duality and G is given by (7), for each $(y, z) \in Y \times Z$, $G(y, z, \cdot)$ is nonincreasing, l.s.c. and such that $G(y, z, +\infty) = -\infty$. Conversely, for any function G satisfying these properties, formula (6) defines a duality ([42, Thm 3.2]) since whenever $f = \inf_{i \in I} f_i$ one has $G(y, z, f(y)) = \sup_{i \in I} G(y, z, f_i(y))$, hence $f^D = \sup_{i \in I} f_i^D$ by passing to the supremum over $y \in Y$ and exchanging the suprema.

When D is a duality, relation (7) defines the unique generating function associated with D. In the general case of a quasi-duality, there is a greatest generating function associated with D obtained by taking the supremum of all generating functions associated with D.

When D is a conjugacy with coupling function c, formula (7) takes a simple form:

$$G(y, z, r) = -(r - c(y, z)). \tag{8}$$

Even in the general case of a quasi-duality, we can get a form of symmetry.

Lemma 5 *Any quasi-duality $D : \overline{\mathbb{R}}^Y \to \overline{\mathbb{R}}^Z$ with generating function G gives rise to a hemi-duality: a reverse mapping E satisfying $E(D(f)) \leq f$ for all $f \in \overline{\mathbb{R}}^Y$ is obtained by using the generating function G' given by*

$$G'(z, y, s) := \inf \left\{r \in \overline{\mathbb{R}} : G(y, z, r) \leq s\right\} \tag{9}$$

by setting

$$E(g)(y) := \sup_{z \in Z} G'(z, y, g(z)). \tag{10}$$

Moreover E is a duality.

When D is a duality, E is the reverse duality D' described by relation (5).

Proof Given $f \in \overline{\mathbb{R}}^Y$, by defn of $f^D := D(f)$, for every $y \in Y, z \in Z$, we have $G(y, z, f(y)) \leq f^D(z)$, hence $G'(z, y, f^D(z)) \leq f(y)$. Therefore the map E associated with G' satisfies

$$E(f^D)(y) := \sup_{z \in Z} G'(z, y, f^D(z)) \leq f(y).$$

The second assertion follows from the fact that, for every $(z, y) \in Z \times Y$ and every family $(s_i)_{i \in I}$, the function $G'(z, y, \cdot)$ is nonincreasing and satisfies $G'(y, z, +\infty) = -\infty$, $G'(z, y, \inf_{i \in I} s_i) = \sup_{i \in I} G'(z, y, s_i)$: setting for $t \in \mathbb{R}$, $R(t) := \{r \in \overline{\mathbb{R}} : G(y, z, r) \leq t\}$ and observing that $R(s)$ is the intersection over $i \in I$ of the intervals $R(s_i)$, one get $\inf R(s) = \sup_{i \in I} \inf R(s_i)$. Thus G' generates a duality.

The last assertion is a consequence of the uniqueness property in Proposition 3 since, when D is a duality, for any given $g \in \overline{\mathbb{R}}^Z$, one has $D(E(g)) \leq g$: given $y \in Y, z \in Z$, setting $R(y, z) := \{r : G(y, z, r) \leq g(z)\}$, $f := E(g)$, we have

$$f(y) \geq G'(z, y, g(z)) = \inf R(y, z),$$

hence $G(y, z, f(y)) \leq G(y, z, G'(z, y, g(z))) = \sup\{G(y, z, r) : r \in R(y, z)\} \leq g(z)$, so that $D(f)(z) \leq g(z)$. □

The framework we intend to use is introduced in the next definition which brings some flexibility.

Definition 6 Given sets Y, Z and a base point 0_Z in Z, a pseudo-duality between the two lattices $\mathcal{Y} := \overline{\mathbb{R}}^Y$ and $\mathcal{Z} := \overline{\mathbb{R}}^Z$ is a triple (A, D, G) such that $D : \mathcal{Y} \to \mathcal{Z}$ is a quasi-duality with generating function G and $A : \mathcal{Z} \to \mathcal{Y}$ is an antitone map such that

$$D(A(g))(0_Z) \leq g(0_Z) \quad \text{for all } g \in \mathcal{Z}. \tag{11}$$

In the sequel, by an abuse of notation, D often stands for the triple (A, D, G). The antitone map A is called the *ante-duality* of D. We observe that not all quasi-dualities are pseudo-dualities as in general one cannot take for A the reverse duality D' which satisfies $D' \circ D \leq I_{\mathcal{Y}}$, with $I_{\mathcal{Y}}$ the identity mapping of $\mathcal{Y}$, but not the relation $D \circ A \leq I_{\mathcal{Z}}$ nor the required relation (11). However, every duality is a pseudo-duality when one takes $A = D'$.

A standard way to get a pseudo-duality is described in the following statement whose proof is obvious (under its assumptions one even has $D \circ A \leq I_{\mathcal{Z}}$).

Lemma 7 *Let $A : \overline{\mathbb{R}}^Z \to \overline{\mathbb{R}}^Y$ be a quasi-duality with generating function F and let $D := A'$ be the reverse map given by Proposition 3. Then the triple (A, D, F') is a pseudo-duality, F' being given as in (9):*

$$F'(y, z, s) := \inf\{r \in \overline{\mathbb{R}} : F(z, y, r) \leq s\}.$$

Let us recall from Proposition 3 that when $D \circ A \leq I_{\mathcal{Z}}$, then, one has $D' \leq A$.

Since for a pseudo-duality the base point 0_Z of Z plays a special role, let us look for conditions on the generating function G of a quasi-duality D which enable to get a pseudo-duality. Let us say that a quasi-duality D (or rather its generating function G) is *quasi-pointed* if it satisfies the assumption

$$\text{(Q)} \qquad G(y, 0_Z, r) \leq -r \quad \text{for all} \quad r \in \overline{\mathbb{R}}, y \in Y$$

and that it is *pointed* if it satisfies the assumption

$$\text{(P)} \qquad G(y, 0_Z, r) = -r, \quad G'(0_Z, y, s) \geq -s \quad \text{for all} \quad r, s \in \overline{\mathbb{R}}, y \in Y.$$

Condition (Q) is satisfied whenever $c(y, 0_Z) \leq 0$ for all $y \in Y$ as then G is given by relation (8): $G(y, z, r) = -(r - c(y, z))$. In particular, the conjugacy associated with a polarity is always quasi-pointed.

Condition (P) is satisfied if D is the conjugacy associated with a coupling function $c : Y \times Z \to \overline{\mathbb{R}}$, satisfying

$$\text{(S)} \qquad c(y, 0_Z) = 0 \quad \text{for all} \quad y \in Y,$$

Let us note that (S) is equivalent to

$$\text{(P}'\text{)} \qquad \sup_{y \in Y} G(y, 0_Z, f(y)) = \sup_{y \in Y} -f(y) \quad \text{for all} \quad f \in \overline{\mathbb{R}}^Y,$$

while (Q) is equivalent to the inequality

$$\text{(Q}'\text{)} \qquad \sup_{y \in Y} G(y, 0_Z, f(y)) \leq \sup_{y \in Y} -f(y) \quad \text{for all} \quad f \in \overline{\mathbb{R}}^Y.$$

Let us also observe that in (P) we have in fact $G'(0_Z, y, s) = -s$ since by relations (9) and (Q), for all $y \in Y, s \in \overline{\mathbb{R}}$ we get

$$G'(0_Z, y, s) \leq -s.$$

The following statement shows that condition (P) enables to reach our aim.

Proposition 8 *If D, or rather (D, G), is a pointed quasi-duality from $\overline{\mathbb{R}}^Y$ to $\overline{\mathbb{R}}^Z$, then (E, D, G) is a pseudo-duality, $E : \overline{\mathbb{R}}^Z \to \overline{\mathbb{R}}^Y$ being given by relations (9) and (10).*

Proof Given $g \in \overline{\mathbb{R}}^Z$, for $f := E(g)$, we get, by (P),

$$\inf_{y \in Y} f(y) := \inf_{y \in Y} E(g)(y) := \inf_{y \in Y} \sup_{z \in Z} G'(z, y, g(z)) \geq \inf_{y \in Y} G'(0_Z, y, g(0_Z)) \geq -g(0_Z)$$

hence

$$D(E(g))(0_Z) := \sup_{y \in Y} G(y, 0_Z, f(y)) \leq \sup_{y \in Y} -f(y) \leq g(0_Z).$$

□

Let us give a generalization of the Fenchel-Moreau Theorem to quasi-dualities. For such a purpose, let us introduce the family $A_G := \{a_{y,r} := G(y, \cdot, r) : y \in Y,\ r \in \mathbb{R}\}$ of *G-affine functions* (or generalized affine functions if there is no ambiguity about G) on Z. A function $g \in \overline{\mathbb{R}}^Z$ which can be represented as the supremum of a family $A_g \subset A_G$ will be called *G-convex*; and we write $g \in \Gamma_G(Z)$ or $\Gamma_D(Z)$ by an abuse of notation (which is justified when D is a duality):

$$g \in \Gamma_G(Z) \Leftrightarrow \exists\, A_g \subset A_G :\ g = \sup\{a : a \in A_g\}.$$

Of course, we can take $A_g := A_G(g) := \{a \in A_G : a \leq g\}$. Similar defns and notation can be given for the space Y. The Fenchel-Moreau theorem relates two ways of introducing G-convex functions.

Theorem 9 *The image of a quasi-duality $D : \overline{\mathbb{R}}^Y \to \overline{\mathbb{R}}^Z$ is contained in the set $\Gamma_G(Z)$ of G-convex functions and coincides with it if D is a duality. The image of $E : \overline{\mathbb{R}}^Z \to \overline{\mathbb{R}}^Y$ coincides with $\Gamma_{G'}(Y)$.*

If D is a duality, for any function $g \in \overline{\mathbb{R}}^Z$, the biconjugate $g^{D'D}$ coincides with the A_G-convex hull $\mathrm{co}_{A_G}(g)$ *of g given by:*

$$\mathrm{co}_{A_G}(g)(z) = \sup\{a_{y,r}(z) : a_{y,r} \in A_G,\ a_{y,r} \leq g\} \quad (z \in Z). \tag{12}$$

In particular, if $g \in \Gamma_G(Z)$ then one has $g = g^{D'D}$.

Of course, when D is a duality, similar results hold by interchanging the roles of Y and Z, G' replacing G.

Proof Clearly, if $g := D(f)$ for some $f \in \overline{\mathbb{R}}^Y$, we have $g = \sup_{y \in Y} G(y, \cdot, f(y)) \in \Gamma_G(Z)$. Conversely, suppose D is a duality and let $g \in \Gamma_G(Z)$, so that there exists a family $((y_i, r_i))_{i \in I}$ such that $g = \sup_{i \in I} G(y_i, \cdot, r_i)$. For $y \in Y$, let us set $I(y) := \{i \in I : y_i = y\}$, $f(y) := \inf\{r_i : i \in I(y)\}$, with the usual convention that $\inf \varnothing = +\infty$. Then, I being the union over $y \in Y$ of the sets $I(y)$, one gets $g = D(f)$ since for all $z \in Z$ one has

$$g(z) = \sup_{y \in Y} \sup_{i \in I(y)} G(y_i, \cdot, r_i) = \sup_{y \in Y} G(y, z, f(y)).$$

The assertion about the image of E stems from the fact that E is a duality.

Now suppose D is a duality and let $g \in \overline{\mathbb{R}}^Z$. Since $g \geq g^{D'D} = \sup_{y \in Y} G(y, \cdot, g^{D'}(y))$, setting $r(y) := g^{D'}(y)$, we have $a_{y,r(y)} \leq g$ and $\sup_{y \in Y} a_{y,r(y)} = g^{D'D}$, hence $g^{D'D} \leq \sup\{a : a \in A_G(g)\}$, with $A_G(g) := \{a \in A_G : a \leq g\}$, as above. On the other hand, if $a := a_{y,r} \in A_G(g)$, by defn of G', for all $z \in Z$ we have $G'(z, y, g(z)) \leq r$ since $G(y, z, r) \leq g(z)$. Thus, $g^{D'}(y) := \sup_{z \in Z} G'(z, y, g(z)) \leq r$. Since $G(y, z, \cdot)$ is nonincreasing, it follows that $a_{y,r}(z) := G(y, z, r) \leq G(y, z, g^{D'}(y)) \leq g^{D'D}(z)$. Thus, $\sup\{a : a \in A_G(g)\} \leq g^{D'D}$ and equality holds. In particular, if $g \in \Gamma_G(Z)$, then one has $g = \sup\{a : a \in A_G(g)\} = g^{D'D}$. □

In the sequel we say that $g \in \overline{\mathbb{R}}^Z$ is (exactly) G-*convex at* some $\overline{z} \in Z$ if there exists some $a \in A_G(g)$ such that $a(\overline{z}) = g(\overline{z})$. Of course, if g is G-convex at every point of Z, then g is G-convex.

The following definition extends to quasi-dualities and pseudo-dualities the notion of subdifferential associated with a duality introduced in Ref. [45, Def.1.2]. We refer to that paper for the motivations of such a choice; see also [5,22]. We start with the case of a function defined on Z, as the defn seems to be more natural in that case and as it is what we shall use. Note that here we are making an abuse of language and notation since uniqueness of the generating function G associated with a quasi-duality is not ensured.

Definition 10 Given a quasi-duality D associated with a generating function G, the *D-subdifferential* of $g \in \overline{\mathbb{R}}^Z$ at $\overline{z}$ is defined by

$$\partial^D g(\overline{z}) = \{y \in Y : G(y, \overline{z}, g^{D'}(y)) = g(\overline{z})\}. \tag{13}$$

Similarly, the *D-subdifferential* of $f \in \overline{\mathbb{R}}^Y$ at $\overline{y}$ is defined by

$$\partial^D f(\overline{y}) = \{z \in Z : G'(z, \overline{y}, f^D(z)) = f(\overline{y})\}. \tag{14}$$

If (A, D, G) is a pseudo-duality, the *A-subdifferential* of $g \in \overline{\mathbb{R}}^Z$ at $\overline{z}$ is defined by

$$\partial^A g(\overline{z}) = \{y \in Y : G(y, \overline{z}, g^A(y)) = g(\overline{z})\}. \tag{15}$$

Since $\sup_{y \in Y} G(y, 0_Z, g^A(y)) = g^{AD}(0_Z) \leq g(0_Z)$ in the case of a pseudo-duality, we have the obvious result which follows. It shows that subdifferentials associated with pseudo-dualities are more interesting than subdifferentials associated with quasi-dualities.

Proposition 11 *Let* (A, D, G) *be a pseudo-duality, let* $0_Z \in Z$ *and let* $g \in \overline{\mathbb{R}}^Z$. *Then, every* $\overline{y} \in \partial^A g(0_Z)$ *is a maximizer of the function* $G(\cdot, 0_Z, g^A(\cdot))$ *and* g *is* G*-convex at* 0_Z. *Conversely, if* g *is* G*-convex at* 0_Z*, then every maximizer of the function* $G(\cdot, 0_Z, g^A(\cdot))$ *is in* $\partial^A g(0_Z)$.

3 Sub-Lagrangians and multipliers

The Lagrangian theory is quite simple, and it does not use the apparatus of duality expounded above. However, we will see that a natural means to get Lagrangians is to use dualities or pseudo-dualities. Moreover, the idea of using abstract convexity in the study of Lagrangians has been extensively exploited ([5,17,24,25,28,32,33,37,46,47,51,56,57,63,65,69,78]...). Here we closely follow [68]. Let us consider the constraint optimization problem

$$(\mathcal{P}) \qquad \text{Minimize} \quad f(x) \qquad x \in F,$$

where F is some nonempty subset of a set X and $f : X \to \mathbb{R}_\infty := \mathbb{R} \cup \{+\infty\}$ is finite at some point of the feasible set F. This problem is equivalent to the unconstrained minimization of $f_F := f + \iota_F$ on X, where ι_F is the indicator function of F. Because f_F is difficult to deal with, it may be of interest to replace it by a family $(\ell_y)_{y \in Y}$ of simpler minorants. Then, setting $L(x, y) := L_y(x) := \ell_y(x)$, one disposes of the estimate

$$\sup_{y \in Y} \inf_{x \in X} \ell_y(x) \leq \inf_{x \in X} f_F(x)$$

which follows from the inequality

$$\sup_{y \in Y} \inf_{x \in X} L(x, y) \leq \inf_{x \in X} \sup_{y \in Y} L(x, y) \tag{16}$$

valid for any function $L : X \times Y \to \overline{\mathbb{R}}$. When the computation of the infimum of ℓ_y is simple enough, one gets an estimate of the value of $(\mathcal{P})$ by solving the *Lagrangian dual problem*

$$(\mathcal{D}_L) \qquad \text{maximize} \quad d_L(y) \quad y \in Y,$$

where the *Lagrangian dual function* d_L is defined by

$$d_L(y) := \inf_{x \in X} L(x, y). \tag{17}$$

In Ref. [68] the bifunction $L : X \times Y \to \overline{\mathbb{R}}$ is called a *sub-Lagrangian* of problem $(\mathcal{P})$. When $f_F(x) = \sup_{y \in Y} L(x, y)$ for all $x \in X$ one says that L is a *Lagrangian* of problem $(\mathcal{P})$. This requirement is not necessary to get the *weak duality* inequality

$$\sup(\mathcal{D}_L) \leq \inf(\mathcal{P}) \tag{18}$$

observed above. One says that there is *no duality gap* when this inequality is an equality. This property may occur for a sub-Lagrangian which is not a Lagrangian. It always occurs when some multiplier is available. Here, the notion of multiplier we adopt is a global notion, not an infinitesimal one like in Ref. [52].

Definition 12 An element $\overline{y}$ of Y is called a *multiplier* (for the sub-Lagrangian L) if $\inf_{x\in X} L(x, \overline{y}) = \inf_{x\in F} f(x)$, or, in other terms, if $d_L(\overline{y}) = \inf f_F(X)$.

The set of multipliers will be denoted by M (or M_L if one needs to stress the dependence on L). A number of studies have been devoted to obtaining conditions ensuring the no gap property or the existence of multipliers (see [5,17,32,37,51] for instance). Some forms of convexity or quasi-convexity implying an infsup property is often involved (see [2,4,14,47,49,80]...). A prototype of such results is the Sion-Von Neumann minimax theorem which represents a noticeable step outside the realm of convex analysis. The very defn of a multiplier leads to the following obvious observation.

Proposition 13 *The set M of multipliers coincides with the set S^*_L of solutions to the dual problem $(\mathcal{D}_L)$ whenever there is no duality gap. This occurs when M is nonempty.*

Thus, when a multiplier is available, one gets the value of problem $(\mathcal{P})$ by solving the unconstrained problem

$$(\mathcal{Q}_{\overline{y}}) \qquad \text{minimize } L(x, \overline{y}) \quad x \in X$$

which is easier to solve than $(\mathcal{P})$ in general. In fact, much more can be expected, as shown in the following statement.

Proposition 14 *([68, Prop. 1.2]) Suppose L is a sub-Lagrangian of $(\mathcal{P})$. If $\overline{x} \in X$ belongs to the set S of solutions to $(\mathcal{P})$ and if $\overline{y} \in M$, then $\overline{x}$ belongs to the set $S_{\overline{y}}$ of solutions to $(\mathcal{Q}_{\overline{y}})$ and $L(\overline{x}, \overline{y}) = f_F(\overline{x})$. Conversely, given $\overline{y} \in Y$, if $\overline{x} \in S_{\overline{y}}$ and if $L(\overline{x}, \overline{y}) = f_F(\overline{x})$, then $\overline{x} \in S$ and $\overline{y} \in M$.*

When the assumptions of the preceding statement are satisfied and when $L(\overline{x}, \overline{y})$ is finite, one can show that $(\overline{x}, \overline{y}) \in X \times Y$ is a *saddle-point* of L on $X \times Y$ in the sense that for any $(x, y) \in X \times Y$ one has

$$L(\overline{x}, y) \leq L(\overline{x}, \overline{y}) \leq L(x, \overline{y}). \tag{19}$$

It is well-known that multipliers may exist while the primal problem $(\mathcal{P})$ has no solution.

It has been observed in Ref. [68] that the situation is symmetric: turning the dual problem $(\mathcal{D}_L)$ into a minimization problem

$$(\mathcal{P}^*_L) \qquad \text{minimize } -d_L(y) \quad y \in Y$$

called the *adjoint problem*, one can note that $-L^T$ given by $-L^T(y, x) := -L(x, y)$ for $(y, x) \in Y \times X$ is a Lagrangian of $(\mathcal{P}^*_L)$ since $\sup_{x\in X} -L(x, y) = -d_L(y)$ for any $y \in Y$ and the *bi-adjoint problem*

$$(\mathcal{P}^{**}_L) \qquad \text{minimize} \quad -d_{LL}(x) := \sup_{y\in Y} L(x, y) \quad x \in X$$

coincides with $(\mathcal{P})$ when L is a Lagrangian. In the general case $(\mathcal{P}^{**}_L)$ may be simpler than $(\mathcal{P})$ and its value is such that

$$\sup(\mathcal{D}_L) \leq \inf(\mathcal{P}^{**}_L) \leq \inf(\mathcal{P}).$$

One always has $(\mathcal{P}^{***}_L) = (\mathcal{P}^*_L)$. The following statement is a direct application of the preceding proposition to the adjoint problem.

Corollary 15 *When* $\sup(\mathcal{D}_L) = \inf(\mathcal{P}_L^{**})$, *the multipliers of the adjoint problem* $(\mathcal{P}_L^*)$ *are the solutions of* $(\mathcal{P}_L^{**})$. *In particular, when L is a Lagrangian and when there is no duality gap, the multipliers of the adjoint problem* $(\mathcal{P}_L^*)$ *are the solutions of* $(\mathcal{P})$.

An illustration of the Lagrangian theory is offered by the familiar case of linear (conical) programming, in which X and Z are normed vector spaces endowed by the preorders $\geq_B$, $\geq_C$ induced by closed convex cones B and C respectively, f is a continuous linear form c on X and the feasible set is given by

$$F := \{x \in X : x \geq_B 0,\ Ax \geq_C b\}$$

where A is a continuous linear map from X to Z and $b \in Z$. Taking $Y := Z^*$ endowed with the order induced by the dual cone $C^{\oplus} := \{y \in Y : \forall z \in C\ \langle y, z\rangle \geq 0\}$ of C, the Lagrangian L is given by

$$\begin{array}{ll} L(x, y) := \langle c, x\rangle + \langle y, b - Ax\rangle & \text{for } (x, y) \in B \times C^{\oplus}, \\ L(x, y) := +\infty & \text{for } (x, y) \in (X\backslash B) \times C^{\oplus}, \\ L(x, y) := -\infty & \text{for } (x, y) \in B \times (Y\backslash C^{\oplus}). \end{array}$$

Then, the dual problem turns out to be of a similar form:

$$(\mathcal{D}_L) \quad \text{maximize} \quad \langle y, b\rangle \quad A^T y \geq_{B^{\oplus}} c,\ y \geq_{C^{\oplus}} 0,$$

where $B^{\oplus}$ is the dual cone of B.

4 Sub-perturbations

As is well known, another device to get duality results is the theory of perturbations. It derives from the observation that many optimization problems are not isolated but are part of families depending on some parameters. As an example, a firm wanting to maximize its profit (or minimize its losses) may take into account the evolution of prices on the market or perform some changes in the production capacity or in the number of employees. As in Ref. [68] we give a one-sided character to the theory.

The perturbational approach assumes that one disposes of a parameter space Z and of a function $P : X \times Z \to \overline{\mathbb{R}}$ which represents a perturbation of the given problem in the sense that for some base point 0_Z of the parameter space Z one has $P(x, 0_Z) = f_F(x)$ for all $x \in X$, where as above, f_F is the extended objective function given by $f_F := f + \iota_F$. It has been observed in Ref. [68] that this condition can be relaxed by just requiring that the function $P : X \times Z \to \overline{\mathbb{R}}$ is a *sub-perturbation of* $(\mathcal{P})$, i.e. is such that $P(x, 0_Z) \leq f_F(x)$ for all $x \in X$. One associates to P the *performance* (or value) function p given by

$$p(z) = \inf_{x \in X} P(x, z). \tag{20}$$

Suppose there exists a pseudo-duality $D : \overline{\mathbb{R}}^Y \to \overline{\mathbb{R}}^Z$ with generating function G on $Y \times Z \times \overline{\mathbb{R}}$ and ante-duality A. The expression of the conjugate p^{AD} of p^A using the generating function G of D leads to the *perturbational dual problem*

$$(\mathcal{D}_P) \qquad \text{maximize} \ \ d_P(y) := G\left(y, 0_Z, p^A(y)\right) \quad \text{over} \quad y \in Y.$$

In fact, one has

$$\sup_{y \in Y} d_P(y) = \sup_{y \in Y} G(y, 0_Z, p^A(y)) = p^{AD}(0_Z) \leq p(0_Z)$$

and one gets the weak duality inequality

$$\sup(\mathcal{D}_P) \leq \inf(\mathcal{P}). \tag{21}$$

When D is just a quasi-duality, in $(\mathcal{D}_P)$ we replace the function p^A by $p^{D'}$. Then, the duality gap $\inf(\mathcal{P}) - \sup(\mathcal{D}_P)$ is not necessarily a nonnegative number. This is the reason why we prefer to use pseudo-dualities rather than quasi-dualities. On the other hand, using Lemma 7, one can use a quasi-duality $A : \overline{\mathbb{R}}^Z \to \overline{\mathbb{R}}^Y$ with generating function F and take for D the reverse quasi-duality associated with $G := F'$.

Since by defn $\partial^A p(0_Z)$ is the set of $\overline{y} \in Y$ such that $G\left(\overline{y}, 0_Z, p^A(\overline{y})\right) = p(0_Z)$, we see that $\overline{y} \in \partial^A p(0_Z)$ if, and only if, $d_P(\overline{y}) = p(0_Z)$, and when $p(0_Z) = \inf(\mathcal{P})$, in particular when P is a perturbation, if and only if, there is no duality gap and $\overline{y}$ is a solution to $(\mathcal{D}_P)$. When D is just a quasi-duality, only part of this assertion is valid: we still have

$$\overline{y} \in \partial^D p(0_Z) \Longleftrightarrow d_P(\overline{y}) = p(0_Z).$$

If D is pointed, the objective d_P of $(\mathcal{D}_P)$ satisfies

$$d_P(y) = -p^A(y). \tag{22}$$

When D is the conjugacy associated to a coupling function c, the objective d_P of $(\mathcal{D}_P)$ can be written

$$d_P(y) = -\left(p^c(y) - c(y, 0_Z)\right). \tag{23}$$

Moreover $d_P(y) = c(y, 0_Z) - p^c(y)$ when $c(\cdot, 0_Z)$ is a finitely valued function; in particular $d_P(y) = -p^c(y)$ when $c(\cdot, 0_Z) = 0$.

Transforming the problem $(\mathcal{D}_P)$ into a minimization problem, we get the perturbational adjoint problem

$$(\mathcal{P}_P^*) \qquad \text{minimize } -G(y, 0_Z, p^A(y)) \quad y \in Y.$$

Again, a certain symmetry appears, since one can associate to that problem a natural perturbation $Q : Y \times Z \to \overline{\mathbb{R}}$ given by $Q(y, z) := -G(y, z, p^A(y))$. Then, the performance function q associated with this perturbation is just $-p^{AD}$: for all $z \in Z$ one has

$$q(z) := \inf_{y \in Y} Q(y, z) = \inf_{y \in Y} -G(y, z, p^A(y)) = -p^{AD}(z).$$

Thus, one has $-q(z) \leq p(z)$ for all $z \in Z$.

Question: Under which conditions can one relate the dual problem of $(\mathcal{P}_P^*)$ to $(\mathcal{P})$?

A problem related to $(\mathcal{P})$ is obtained by considering the sub-perturbation $R(x, z) := P_x^{AD}(z)$, where $P_x := P(x, \cdot)$, and its associated performance function: $r(z) := \inf_{x \in X} R(x, z)$. Since E is a duality and $E \circ (D \circ A) = (E \circ D) \circ A \leq A$, the conjugate r^E of r is such that

$$r^E = (\inf_{x \in X} P_x^{AD})^E = \sup_{x \in X} P_x^{ADE} \leq \sup_{x \in X} P_x^A \leq (\inf_{x \in X} P_x)^A = p^A.$$

Thus, the objective $G(\cdot, 0_Z, r^E(\cdot))$ of the dual problem associated to the perturbation R is greater than or equal to the objective $G(\cdot, 0_Z, p^A(\cdot))$ of $(\mathcal{D}_P)$. When A is a duality with reverse mapping D and when P_x is A-convex at each point for all $x \in X$, the dual problem associated with R coincides with the one associated with P.

5 Passages between the two approaches

Let us try to compare the two approaches in the general framework we have adopted. In order to do so, we assume a pseudo-duality $(A, D, G) := D : \overline{\mathbb{R}}^Y \to \overline{\mathbb{R}}^Z$ is given and a base point 0_Z has been chosen in Z.

5.1 From perturbations to Lagrangians

Given a sub-perturbation $P : X \times Z \to \overline{\mathbb{R}}$ of $(\mathcal{P})$ and $x \in X$, we denote by P_x the function $P(x, \cdot)$ and by $P_x^A := (P_x)^A$ its conjugate function; similarly, for a function $L : X \times Y \to \overline{\mathbb{R}}$ we set $L_x := L(x, \cdot)$.

Proposition 16 *Let P be a sub-perturbation of $(\mathcal{P})$, let (A, D, G) be a pseudo-duality with generating function G. Then the function L given by*

$$L(x, y) := G(y, 0_Z, P_x^A(y)) \qquad x \in X,\ y \in Y, \tag{24}$$

is a Lagrangian of the relaxed problem $(\mathcal{P}^{AD})$ consisting in minimizing over X the function $x \mapsto P_x^{AD}(0_Z)$; thus L is a sub-Lagrangian for $(\mathcal{P})$. When P is a perturbation and when $P_x^{AD}(0_Z) = P_x(0_Z)$ for all $x \in X$, L is a Lagrangian for $(\mathcal{P})$.

Moreover, the objective functions of d_P and d_L of the dual problems associated with P and L, respectively satisfy $d_P \le d_L$ and one has $\sup d_P \le \sup d_L \le p(0_Z)$. When A is a duality and $G(y, 0_Z, \cdot)$ is upper semicontinuous for all $y \in Y$, in particular when D is a pointed duality, the two dual problems have the same objective functions, hence the same values and the same sets of solutions.

The inequality $d_L \ge d_P$ ensures that the dual problem is not deteriorated when passing from P to L. Thus, when there is no duality gap for $(\mathcal{D}_P)$, there is no duality gap for $(\mathcal{D}_L)$.

When D is pointed, the Lagrangian L takes the simpler form

$$L_x = -P_x^A\ . \tag{25}$$

When D is the conjugacy associated with a coupling function c, since $G(y, z, r) = -(r - c(y, z))$ for $(x, y, r) \in X \times Y \times \overline{\mathbb{R}}$, one has

$$L(x, y) = -(P_x^A(y) - c(y, 0_Z)) \quad \forall (x, y) \in X \times Y.$$

Note that L is not necessarily a Lagrangian for $(\mathcal{P})$, even when P is a perturbation of $(\mathcal{P})$.

Proof The first assertions are consequences of the following relations

$$\sup_{y \in Y} L(x, y) = \sup_{y \in Y} G(y, 0_Z, P_x^A(y)) = P_x^{AD}(0_Z) \le P_x(0_Z) \le f(x). \tag{26}$$

Since for all $x \in X$, $y \in Y$ one has $P_x \ge p$, hence $P_x^A(y) \le p^A(y)$ and since $G(y, 0_Z, \cdot)$ is nonincreasing, one has

$$d_L(y) := \inf_{x \in X} L(x, y) = \inf_{x \in X} G(y, 0_Z, P_x^A(y)) \ge G(y, 0_Z, p^A(y)) = d_P(y),$$

hence $\sup_{y \in Y} d_L(y) \ge \sup_{y \in Y} d_P(y)$. On the other hand,

$$\begin{aligned} \sup_{y \in Y} d_L(y) &:= \sup_{y \in Y} \inf_{x \in X} L(x, y) \le \inf_{x \in X} \sup_{y \in Y} G(y, 0_Z, P_x^A(y)) \\ &= \inf_{x \in X} P_x^{AD}(0_Z) \le \inf_{x \in X} P_x(0_Z) = p(0_Z). \end{aligned}$$

Suppose $G(y, 0_Z, \cdot)$ is upper semicontinuous and A is a duality (it is the case when A is the reverse map E given by Proposition 3). Then, one has

$$d_L(y) := \inf_{x \in X} G(y, 0_Z, P_x^A(y)) = G(y, 0_Z, \sup_{x \in X} P_x^A(y)) = G(y, 0_Z, p^A(y)) = d_P(y).$$

□

In the following corollary which follows from (26), we recover a classical statement.

Corollary 17 *If P is a perturbation of $(\mathcal{P})$, if D is a duality and if for all $x \in X$ the function P_x is D-convex at 0_Z, then the function L given by (24) is a Lagrangian for $(\mathcal{P})$.*

In the next result we suppose L is the sub-Lagrangian associated with a sub-perturbation P and a pseudo-duality (A, D, G) and we compare the set M of multipliers of L with $\partial^A p(0_Z)$.

Proposition 18 *Let P be a sub-perturbation of $(\mathcal{P})$ and let L be the sub-Lagrangian associated with P as in (24). Then, $M \subset \partial^A p(0_Z)$ and one has $p(0_Z) = \inf(\mathcal{P})$ if M is nonempty. When $p(0_Z) = \inf(\mathcal{P})$ one has*

$$\partial^A p(0_Z) = M.$$

Proof Let $\overline{y} \in M$. Then, by Definition 12, Propositions 16 and 13, one has $\inf(\mathcal{P}) = \sup(\mathcal{D}_L) \leq p(0_Z) \leq \inf(\mathcal{P})$ and equality holds. Moreover, one has $d_P(\overline{y}) := G(\overline{y}, 0_Z, p^A(\overline{y})) = p(0_Z)$, hence $\overline{y} \in \partial^A p(0_Z)$.

Now suppose $p(0_Z) = \inf(\mathcal{P})$ and let $\overline{y} \in \partial^A p(0_Z)$. Then,

$$p(0_Z) \geq d_L(\overline{y}) \geq d_P(\overline{y}) := G(\overline{y}, 0_Z, p^A(\overline{y})) = p(0_Z)$$

so that $d_L(\bar{y}) = p(0_Z) = \inf(\mathcal{P})$: one gets $\overline{y} \in M$.

Remark 1 (a) The preceding result can also be deduced from the comparison made above between $\partial^A p(0_Z)$ and the solution set S_P^* to $(\mathcal{D}_P)$.

(b) The equality $\inf(\mathcal{P}) = p(0_Z)$ may occur for a sub-perturbation which is not a perturbation.

□

5.2 From Lagrangians to perturbations

We have detected some conditions ensuring that a perturbational dual problem can be considered as a Lagrangian dual problem. Now let us tackle the question: is there is a reverse passage? Of course, as mentioned in the beginning of the section, we have to suppose a pseudo-duality is given.

Proposition 19 *Let (A, D, G) be a pseudo-duality, where $A := E$, the map given by (9) and (10), let $L : X \times Y \to \overline{\mathbb{R}}$ be a sub-Lagrangian of $(\mathcal{P})$ and let P, p, d_P be defined by the following formulas*

$$P(x, z) := (-L_x)^D(z), \qquad p(z) := \inf_{x \in X} P(x, z), \qquad d_P(y) := G(y, 0_Z, p^A(y)), \tag{27}$$

where $L_x = L(x, \cdot)$. Then, one has $\sup_{y \in Y} d_P(y) \geq (-d_L)^D(0_Z)$. When D is quasi-pointed, P is a sub-perturbation of $(\mathcal{P})$, in fact, a sub-perturbation of the relaxed problem of minimizing over X the function $x \mapsto k(x) := \sup_{y \in Y} L_x(y)$. Moreover, when D is pointed, one has $d_P = d_K \geq d_L$, where K is given by $K(x, y) := -(-L_x)^{DA}(y)$.

Thus, when D is quasi-pointed, the value of the dual problem is not deteriorated when passing from the Lagrangian dual to the perturbational dual.

Proof Since $A = E$ is a duality, the performance function p of P satisfies

$$p^A(y) = (\inf_{x \in X} P_x)^A(y) = \sup_{x \in X}(P_x)^A(y) = \sup_{x \in X}(-L_x)^{DA}(y) \leq \sup_{x \in X}(-L_x)(y) = -d_L(y). \tag{28}$$

Thus, the value of the dual problem associated to P satisfies

$$\sup_{y \in Y} d_P(y) = \sup_{y \in Y} G(y, 0_Z, p^A(y)) \geq \sup_{y \in Y} G(y, 0_Z, -d_L(y)) = (-d_L)^D(0_Z).$$

When D is quasi-pointed, the fact that P is a sub-perturbation for the minimization of k stems from the relations

$$\forall x \in X \qquad P(x, 0_Z) := \sup_{y \in Y} G(y, 0_Z, -L_x(y)) \leq \sup_{y \in Y} L_x(y) \leq f_F(x)$$

which are equalities when L is a Lagrangian and when D is pointed. When D is pointed, since for all $x \in X$ one has $-K_x := (-L_x)^{DA} \leq -L_x$, using (28), one gets, for any $y \in Y$,

$$\begin{aligned} d_P(y) &:= G(y, 0_Z, p^A(y)) = -p^A(y) = -\sup_{x \in X}(-L_x)^{DA}(y) = \inf_{x \in X} K(x, y) \\ &= d_K(y) \geq d_L(y). \end{aligned}$$

□

The next assertion is a consequence of the relation $K = L$ when $-L_x = (-L_x)^{DA}$ for all $x \in X$. Then equality holds in (28) and in the next relation.

Corollary 20 *Let D be a pointed pseudo-duality. If for all $x \in X$ the function $-L_x := -L(x, \cdot)$ is such that $-L_x = (-L_x)^{DA}$, then the dual problems associated with L and the sub-perturbation P deduced from L coincide: $d_L = d_P$.*

In the next result we suppose L is a given sub-Lagrangian and P is the sub-perturbation associated to L via a pseudo-duality (A, D, G) and we compare the set M of multipliers of L with $\partial^A p(0_Z)$.

Proposition 21 *Let L be a sub-Lagrangian of $(\mathcal{P})$ and let P be the sub-perturbation associated with L as in (27). Then, $M \subset \partial^A p(0_Z)$ and one has $p(0_Z) = \inf(\mathcal{P})$ if M is nonempty. When D is pointed, $-L_x = (-L_x)^{DA}$ for all $x \in X$ and when $p(0_Z) = \inf(\mathcal{P})$ one has*

$$\partial^A p(0_Z) = M.$$

Proof Since D is a pointed duality, we have $\sup d_L \leq \sup d_P \leq p(0_Z)$. Let $\overline{y} \in M$. Then, since P is a sub-perturbation of $(\mathcal{P})$, we have $\inf(\mathcal{P}) = d_L(\overline{y}) \leq d_P(\overline{y}) \leq p(0_Z) \leq \inf(\mathcal{P})$ by Definition 12, Propositions 13 and 19, so that equality holds. Moreover, one has $p(0_Z) = d_L(\overline{y}) = d_P(\overline{y}) := G(\overline{y}, 0_Z, p^A(\overline{y}))$, hence $\overline{y} \in \partial^A p(0_Z)$.

Now suppose D is pointed, $p(0_Z) = \inf(\mathcal{P})$, $-L_x = (-L_x)^{DA}$ for all $x \in X$ and let $\overline{y} \in \partial^A p(0_Z)$. Then, $d_L = d_P$ by the preceding corollary and

$$p(0_Z) \geq d_L(\overline{y}) = d_P(\overline{y}) := G(\overline{y}, 0_Z, p^A(\overline{y})) = p(0_Z)$$

so that $d_L(\overline{y}) = p(0_Z) = \inf(\mathcal{P})$: one gets $\overline{y} \in M$.

Remark 2 (a) The preceding result can also be deduced from the comparison made above between $\partial^A p(0_Z)$ and the solution set S_P^* to $(\mathcal{D}_P)$.
(b) The equality $\inf(\mathcal{P}) = p(0_Z)$ may occur for a sub-perturbation which is not a perturbation.

□

5.3 Iterating the passages

One may wonder what happens when one makes successively the two passages described above assuming one is given a pseudo-duality (A, D, G).

First, assume that D is a pointed pseudo-duality such that $A = E$ and let us start with a sub-Lagrangian L. Then we denote by L^P the sub-Lagrangian associated with the perturbation P deduced from L via relation (27): for any $(x, y) \in X \times Y$, by (25), one has $L^P(x, y) := G(y, 0_Z, P_x^A(y))$, hence $\sup d_{L^P} \geq \sup d_P \geq \sup d_L$: the value of the new dual problem is not deteriorated by this double passage. Moreover, the objective of the new dual problem is not deteriorated by this double passage since $L^P(x, y) = -(-L_x)^{DA}(y) = K(x, y) \geq L(x, y)$.

Now, let us start with a sub-perturbation P, assuming again that D is a pointed pseudo-duality such that $A = E$. Let us denote by P^L the sub-perturbation associated with the sub-Lagrangian L deduced from P via relation (25): for any $(x, z) \in X \times Z$, by (27), one has $P^L(x, z) := (-L_x)^D(z)$, hence $P^L(x, 0_Z) = P_x^{AD}(0_Z) \leq P_x(0_Z)$, so that P^L is again a sub-perturbation of $(\mathcal{P})$ and a perturbation when is a P perturbation and $P_x^{AD} = P_x$.

Thus, when (S) holds, it is possible to characterize the family of sub-perturbations which are obtained from a sub-Lagrangian and the sub-Lagrangians which are obtained from a sub-perturbation via the preceding processes.

Corollary 22 *When D is the duality associated with a conjugacy c and when (S) holds, a sub-perturbation P is obtained from a sub-Lagrangian if, and only if, for each $x \in X$ one has $P_x = P_x^{cc}$ and a sub-Lagrangian L is obtained from a sub-perturbation if, and only if, for each $x \in X$ one has $L_x = -(-L_x)^{cc}$.*

Proof When P is deduced from L by the preceding process, for each $x \in X$ one has $P_x = (-L_x)^c$, hence $P_x^{ccc} = (-L_x)^{ccc} = (-L_x)^c = P_x$. Conversely, when for each $x \in X$ one has $P_x = P_x^{cc}$, setting $L_x := -P_x^c$, one has $P_x = (-L_x)^c$ and P is deduced from L by the preceding process.

The proof of the second assertion is similar. □

6 Duality in composite or constrained optimization

Given sets X, Y, Z, functions $f : X \to \mathbb{R}_\infty$, $h : X \times Z \to \mathbb{R}_\infty$ and a multimap $\Psi : X \rightrightarrows Z$, we consider the problem

$$(\mathcal{C}) \quad \text{minimize } f(x) + h(x, w) \qquad x \in X, w \in \Psi(x).$$

Such a problem arises when considering the minimization of a composite function of the form $x \mapsto h(x, g(x))$ with $f := 0$, $\Psi(\cdot) = \{g(\cdot)\}$. It also appears when minimizing a function $x \mapsto f(x)$ under a constraint of the form $x \in \Phi(C)$ for some multimap $\Phi : Z \rightrightarrows X$, when one takes $\Psi = \Phi^{-1}$, $h(x, z) := \iota_C(z)$, where ι_C is the indicator function of C. In particular, the minimization problem of f under the constraint $g(x) \in C$ for some subset C of Z can be set under this form by taking $\Psi(\cdot) = \{g(\cdot)\}$ and $h(x, z) := \iota_C(z)$.

Given a pseudo-duality (A, D, G) between Y and Z, a sub-Lagrangian L associated to $(\mathcal{C})$ is

$$L(x, y) := f(x) + \inf\{G(y, z, h_x^A(y)) : z \in \Psi(x)\},$$

where $h_x := h(x, \cdot)$. In fact,

$$\begin{aligned}\sup_{y\in Y} L(x, y) &\leq f(x) + \inf_{z\in\Psi(x)} \sup_{y\in Y} G(y, z, h_x^A(y)) \\ &= f(x) + \inf_{z\in\Psi(x)} h_x^{AD}(z) \leq f(x) + \inf_{z\in\Psi(x)} h_x(z).\end{aligned}$$

When Z is endowed with an additive group structure, another way of obtaining a dual problem consists in using the perturbation P given by

$$P(x, z) := f(x) + \inf_{w\in\Psi(x)} h(x, w + z). \tag{29}$$

Assume for simplicity that $h(x, z) := k(z)$ for some $k : Z \to \mathbb{R}_\infty$ and that the pseudo-duality is the conjugacy given a coupling function $c : Y \times Z \to \mathbb{R}$. Then the sub-Lagrangian introduced above is given by

$$L(x, y) = f(x) + \inf_{z\in\Psi(x)} c(y, z) - k^c(y).$$

In particular, when $\Psi(\cdot) := \{g(\cdot)\}$, the Lagrangian L associated with P takes the familiar form $L(x, y) = f(x)+c(y, g(x))-k^c(y)$. It is not difficult to check that, when c is additive in its second variable, L coincides with the sub-Lagrangian associated with the perturbation P given in (29).

We have seen that the passage from P to L is nice when, for all $x \in X$, P_x is c-convex. When Z is a normed vector space, Y is its dual and c is the ordinary coupling, this property is satisfied when k is convex, proper and lower semicontinuous. Under an additional assumption, one gets that p itself is convex. Let us say that (f, g) is *convexlike* (in Ref. [13] it is said that (f, g) is epi-convex) if the set

$$E_{f,g} := \{(z, r) \in Z \times \mathbb{R} : \exists x \in g^{-1}(z), r > f(x)\}$$

is convex; see [20,23,24] and their references for various forms of such a condition. Note that $E_{f,g}$ is the image by the symmetry $(z, r) \mapsto (-z, r)$ of the strict epigraph of the performance function q given by

$$q(z) := \inf\{f(x) : g(x) + z = 0\}.$$

The following result from Ref. [13] makes clear the interest of the assumption of convexlikeness. We present a simple proof.

Lemma 23 *If (f, g) is convexlike, if h is convex, then p is convex. In fact p is the infimal convolution $h\Box q$ of q and h.*

Proof . The result is justified by the following equalities:

$$(h\Box q)(z) = \inf_{w\in Z} (h(w + z) + q(-w)) = \inf_{w\in Z} \inf\{h(g(x) + z) + f(x) : x \in X, g(x) = w\}$$

as the last side is just $p(z)$. □

7 Examples of duality schemes

Let us present various examples showing the versatility of the two approaches.

Example 1 (conjugacies) As mentioned above, given sets Y, Z and a coupling function $c : Y \times Z \to \overline{\mathbb{R}}$, one gets a generating function G by setting $G(y, z, r) := -(r - c(y, z))$. Many dualities using such a simple device, including augmented Lagrangian dualities are displayed in Refs. [61,64,68,71,74,82] and elsewhere.

Example 2 (conjugacies associated with polarities; see [54,55,60,74,82,87]). Given a subset P of $Y \times Z$ considered as a multimap $P : Y \rightrightarrows Z$ with graph P or a polarity $P : 2^Y \to 2^Z$ defined by $P(S) = \bigcap_{y \in S} P(y)$, let us associate the coupling function c given by $c(y, z) := -\iota_Q(y, z)$, where ι_Q is the indicator function of $Q := (Y \times Z)\backslash P$. Then the conjugate of $f : Y \to \overline{\mathbb{R}}$ is given by

$$f^P(z) = \sup\{-f(y) : y \in Y\backslash P^{-1}(z)\} \quad \text{for } z \in Z, \tag{30}$$

and a similar formula holds for the conjugate of a function on Z. Such a conjugacy which represents a versatile tool for the study of various functions appearing in mathematical economics has peculiar properties. The main one is the property that the sublevels of the conjugate of a function f are easily determined.

Proposition 24 *([60,81,87]) For any extended real-valued function f on Y, the conjugate f^P of f is P-quasiconvex in the sense that for each $r \in \mathbb{R}$ the sublevel set $[f^P \leq r]$ is P-convex, i.e. in $P(2^Y)$. More precisely, one has*

$$[f^P \leq r] = P([f < -r]). \tag{31}$$

Example 3 (radiant duality, [53,60,75,83–85,88,89]) In the case Y and Z are n.v.s. in duality via a continuous bilinear form $\langle \cdot, \cdot \rangle$, one can take $P^\wedge := \{(y, z) : \langle y, z \rangle < 1\}$ or $P^\Delta := \{(y, z) : \langle y, z \rangle \leq 1\}$. The corresponding conjugacies are adapted to radiant functions, i.e. quasi-convex functions whose sublevel sets are convex and contain 0. The subdifferential $\partial^\wedge$ associated to $P^\wedge$ is given by

$$\partial^\wedge f(\overline{z}) := \{\overline{y} \in Y : \langle \overline{y}, \overline{z} \rangle \geq 1,\ f(z) \geq f(\overline{z})\ \forall z \in [\overline{y} \geq 1]\}$$

for $f \in \mathbb{R}^Z, \overline{z} \in Z$. The subdifferential associated to P^Δ seems to be of less interest. Associating to P^Δ the ante-duality $P^\wedge$ may be a means to increase its value.

Example 4 (musical and financial quasi-dualities, [1,7–10,59,67,66])Given a normed vector space Y with dual space Y^*, the following conjugates have been studied in several papers. For $f : Y \to \overline{\mathbb{R}}$, $z := (p, q) \in Z := Y^* \times \mathbb{R}$, let $0_Z := (0, 0)$ and

$$\begin{aligned} f^\flat(p, q) &:= \sup\{p.z : z \in [f < q]\}, \\ f^\sharp(p, q) &:= \sup\{p.z : z \in [f \leq q]\}, \\ f^\div(p, q) &:= \sup\{p.z - f(z) : z \in [f < q]\}, \\ f^\%(p, q) &:= \sup\{p.z - f(z) : z \in [f \leq q]\}. \end{aligned}$$

The maps $f \mapsto f^\flat$, $f \mapsto f^\div$ are dualities. The maps $f \mapsto f^\sharp$, $f \mapsto f^\%$ are quasi-dualities, but not dualities. However, they can be taken as natural ante-dualities. Generating functions for these quasi-dualities can be given explicitely by

$$G_\flat(y, p, q, r) = p.y - \iota_{[-\infty,q)}(r), \qquad G_\sharp(y, p, q, r) = p.y - \iota_{[-\infty,q]}(r),$$
$$G_\div(y, p, q, r) = p.y - r - \iota_{[-\infty,q)}(r), \qquad G_\%(y, p, q, r) = p.y - r - \iota_{[-\infty,q]}(r).$$

The reverse generating functions deduced from relation (9) are given by

$$G'_\flat(p,q,y,s) = G'_\sharp(p,q,y,s) = q - \iota_{[-\infty,0]}(s - p.y),$$
$$G'_\div((p,q),y,s) = G'_\%(p,q,y,s) = q \wedge (p.y - s)$$

We observe that these generating functions are quasi-pointed, but not pointed.

Example 5 (integral duality, [2,13]). The duality theory of convex integral functionals has been devised by R.T. Rockafellar about four decades ago. Here we would like to expound some results due to Bourass and Giner in [13] for nonconvex integral functionals and point out their relationships with what precedes.

Recall that if $(S, \mathcal{T}, \mu)$ is a measured space and if E is a separable Banach space, a subset X of the space $L_0(S, E)$ of measurable maps from S to E is said to be *decomposable* if for any $f, g \in X$ and $T \in \mathcal{T}$ one has $\chi_{S \setminus T} f + \chi_T g \in X$, where, for a subset R of S, χ_R denotes the characteristic function of R.

Given a measurable integrand $f : S \times E \to \overline{\mathbb{R}}$, the integral functional $I_f : L_0(S, E) \to \overline{\mathbb{R}}$ is defined by

$$I_f(x) := I(f \circ x) := \int_S f(s, x(s)) d\mu,$$

where the integral is the infimum of the integrals of the elements v of $L_1(S)$ majorizing $f \circ x := f(\cdot, x(\cdot))$. Let us recall the following result which uses the fact that the set $L_0(S)$ of measurable functions on S is a complete inf-lattice: any family $(f_i)_{i \in I}$ of $L_0(S)$ has an infimum $f = L_0 - \inf(f_i)$.

Proposition 25 *([13]) Given a measurable integrand f and a decomposable set X such that* $\inf_{x \in X} I_f(x) < \infty$, *one has*

$$\inf_{x \in X} I_f(x) = \int_S L_0 - \inf_{x \in X} (f \circ x) d\mu.$$

Let X and Y be two decomposable subsets of $L_0(S)$. One says that X is *rich in* Y if for any $y \in Y$ there exists a μ-finite covering (S_n) of S and a sequence (x_n) of X such that $1_{S_n} y = 1_{S_n} x_n$ for all $n \in \mathbb{N}$.

Proposition 26 *Let f be a measurable integrand, $\mathcal{T}$ being μ-complete and let $M : S \rightrightarrows E$ be a measurable multifunction. Then for any decomposable set X which is rich in the set $L_0(M)$ of measurable selections of M one has*

$$\inf_{x \in X} I_f(x) = \int_S \inf_{e \in M(s)} f(s, e) d\mu.$$

This result stems from the fact that

$$L_0 - \inf_{x \in X} (f \circ x)(s) = \inf_{e \in M(s)} f(s, e) \quad \text{a.e.}$$

In the sequel we suppose $\mathcal{T}$ is μ-complete and μ is atomless. Given measurable integrands $f, g_1, ..., g_n : S \times E \to \overline{\mathbb{R}}$, using the Lyapunov convexity theorem one can prove the following result.

Proposition 27 *Let $M : S \rightrightarrows E$ be a measurable multifunction. Then for any decomposable set X which is rich in the set $L_0(M)$ of measurable selections of M, the pair (I_f, I_g) is convexlike on X.*

Let us apply what precedes to the optimization problem

$$(\mathcal{I}) \qquad \text{minimize} \;\; I_f(x) + h(I_g(x)) \quad x \in X,$$

where $f, g := (g_1, ..., g_n)$ are measurable integrands, X is a decomposable subset of $L_0(S, E)$ which is rich in the set $L_0(M)$ of measurable selections of M and $h : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is a convex function. We suppose the value $\inf(\mathcal{I})$ of $(\mathcal{I})$ is finite.

Proposition 28 *Suppose $cl(\mathbb{R}_+(\mathrm{dom}\, h - g(\mathrm{dom}\, f)))$ is a vector subspace of $\mathbb{R}^n$. Then strong duality holds and*

$$\inf(\mathcal{I}) = \max\{I(\ell_M(y)) - h^*(y) : y \in \mathbb{R}^n\},$$

where

$$\ell_M(y)(s) = \inf\{f(s,e) + \langle y, g(s,e)\rangle : e \in M(s) \cap \mathrm{dom}\, g(s,\cdot)\}$$

This result reduces the search of the value of $(\mathcal{I})$ to the solution of a finite dimensional maximization problem. In particular, when h is the indicator function of the negative cone of $\mathbb{R}^n$, a Slater type condition ensures the existence of multipliers. A number of consequences can be drawn from that result, in particular for the nonsmooth analysis of integral functionals (see [26]).

Other examples of duality schemes are given in Refs. [5,16,27,34,36,38,39,41,46,54,56, 68,74–85]for instance.

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